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Continuously variable survival exponent for random walks with movable partial reflectors

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We study a one-dimensional lattice random walk with an absorbing boundary at the origin and a movable partial reflector. On encountering the reflector at site x, the walker is reflected (with probability r) to $x-1$ and the reflector is simultaneously pushed to $x+1$. Iteration of the transition matrix, and asymptotic analysis of the probability generating function show that the critical exponent δ governing the survival probability varies continuously between 1/2 and 1 as r varies between 0 and 1. Our study suggests a mechanism for nonuniversal kinetic critical behavior, observed in models with an infinite number of absorbing configurations.

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Random walks find application in virtually every area of physics [1–5]. Random walks in the presence of traps or absorbing boundaries, and/or reflectors, are much studied as models of exciton recombination [6], diffusion-limited reactions [5], and polymer-surface interactions [7].

In this Rapid Communication, we study an unbiased random walk with an absorbing boundary at the origin, and a movable, partial reflector. At each encounter between walker and reflector, the latter moves one step to the right (i.e., to a site not yet visited by the walker), while the walker is reflected to its previous position with probability r. Thus the reflector hampers the advance of the walker into new territory, but does not otherwise influence its motion. (Note that in the limit $r=0$ the reflector has no effect, but it does mark the *span* of the walk.) We are primarily interested in the effect of the reflector on the asymptotic scaling properties of the walk.

The process defined above admits various physical interpretations. One is in terms of adlayer growth, with deposition and evaporation at the edge of the adlayer (but not in the bulk). The substrate adsorption sites are originally in a "non-activated" state, with a low sticking probability for incident molecules, but after a first adsorption attempt (at a step edge) the substrate site becomes activated, with a high sticking probability. Biological interpretations are also possible, e.g., of the advance of a bacterial colony in a growth medium, with a preliminary contact facilitating expansion into new regions, or, similarly, the spread of a political viewpoint in an initially skeptical population. Our results are of interest as an example of nonuniversality (a continuously variable critical exponent) in a model that allows an exact asymptotic analysis.

A further motivation for our study is provided by absorbing-state phase transitions, an area of great current interest in nonequilibrium statistical physics [8,9], in which a many-particle system, such as directed percolation, becomes trapped in a configuration allowing no further evolution. Continuous transitions to an absorbing state have been invoked in models of epidemics [10], catalysis [11], and the transition to turbulence [12,13], to cite but a few examples. The connection between absorbing-state phase transitions and random walks lies in *compact* directed percolation (CDP) [14]. CDP is defined on a discrete space time (x,t) with time slices corresponding to diagonals of a square lattice, so that the neighbors of site (x,t) at the previous time are $(x-1,t-1)$ and $(x+1,t-1)$. Each site is either occupied or vacant. If $(x,t-1)$ has n occupied neighbors, then (x,t) is vacant (occupied) with probability 1 if $n=0$ (2), and is occupied with probability p if $n=1$. CDP exhibits an absorbing-state phase transition at $p=1/2$. Consider an initial state of a single occupied site in an otherwise empty lattice. The boundaries between the occupied region (descended from the initial seed particle) and the outer vacant regions follow simple random walks, which are unbiased if $p=1/2$. Thus the *length* $X(t)$ of the occupied region, being the distance between two random walks, is also a random walk, with an absorbing boundary at the origin. For $p<1/2$ ($>1/2$), $X(t)$ is attracted to (repelled by) the origin. For $p=1/2$, $X(t)$ is unbiased, and well-known results for random walks [1,4,15] imply that the survival probability decays for long times $\sim t^{-\delta}$, with $\delta=1/2$.

The motivation for introducing mobile reflectors in CDP (and thus in the simple random walk studied here) arises from the puzzling behavior of models that can become trapped in one of an *infinite* number of absorbing configurations (INAC) [16–18]. Anomalies in critical spreading for INAC, such as continuously variable critical exponents, have been traced to a long memory in the dynamics of the order parameter, ρ, due to coupling to an auxiliary field that remains frozen in regions where $\rho=0$ [18,19]. INAC appears to be particularly relevant to the transition to spatiotemporal chaos, as shown in a recent study of a coupled-map lattice with "laminar" and "turbulent" states, which revealed continuously variable spreading exponents [20].

Grassberger, Chaté, and Rousseau (GCR) [21] proposed that spreading in systems with INAC could be understood by studying a model with a *unique* absorbing configuration, but in which the spread of activity to previously inactive regions is hampered (or facilitated). Our model represents a two-fold simplification of the GCR model: first, the appearance of inactive sites within a string of active ones is prohibited (analogous to going from DP to the more restrictive CDP

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process); second, we study a single random walker rather than the pair needed to describe CDP [23]. We believe, nonetheless, that our model captures the essential physics underlying anomalous critical behavior in models with INAC. Continuously variable exponents have been found in DP and directed self-avoiding walks [24], and in CDP [25] confined to *fixed* parabolic geometries.

We study an unbiased, discrete-time random walk on the nonnegative integers, $x_t=0,1,2,\ldots,$ with $x=0$ absorbing. Initially the walker is at $x_0=1$ and the reflector, whose position we denote by R_t, is at $R_0=2$. Each time the walker steps to the site occupied by the reflector, it is reflected back to $R-1$ with probability r (and remains at R with probability $1-r$), while the reflector moves (with probability 1) to $R+1$. Evidently the process x_t is non-Markovian, since the transition probability into a given site depends on whether it has been visited before. We can transform the model to a Markov process by enlarging the state space to include the reflector position; it is convenient to introduce the variable $y_t=R_t-1$ for this purpose. The process (x,y) is restricted to the wedge between $x=0$ (absorbing) and $x=y$, with transitions from y to $y+1$ allowed only from the diagonal $x=y$. At this point it is useful to include a further generalization of our model, by assigning a probability p' for the walker to jump to the right (and $q'=1-p'$ to jump to the left) when on the diagonal; these transition probabilities are summarized in Fig. 1. For $p'<1/2$ the walker experiences an additional impediment to visiting new territory, while for $p'(1-r)>1/2$ the "reflector" effectively becomes an accelerator, drawing the walker forward to a previously unvisited site.

The nonzero transition probabilities for the Markov chain are:

$$\begin{gathered} W[(x,y)\to(x\pm 1,y)]=1/2, \quad x=1,\ldots,y-1, \\ W[(y,y)\to(y,y+1)]=p'r, \\ W[(y,y)\to(y+1,y+1)]=p'(1-r), \\ W[(y,y)\to(y-1,y)]=1-p'. \end{gathered} \tag{1}$$

The probability $P(x,y,t)$ follows the master equation

$$\begin{gathered} P(x,y,t+1)=\tfrac{1}{2}P(x-1,y,t)+\tfrac{1}{2}P(x+1,y,t), \\ x=1,2,\ldots,y-2; \quad y\geqslant 3, \end{gathered} \tag{2}$$

with $P(0,y,t)=0$, representing the absorbing boundary at $x=0$. Letting $D(y,t)\equiv P(y,y,t)$, the boundary conditions along the diagonal are

$$\begin{aligned} P(y-1,y,t+1)=&\tfrac{1}{2}P(y-2,y,t)+p'rD(y-1,t) \\ &+(1-p')D(y,t), \end{aligned} \tag{3}$$

$$D(y,t+1)=\tfrac{1}{2}P(y-1,y,t)+p'(1-r)D(y-1,t),$$

for $y\geqslant 3$. For $y=1$, 2 the evolution equations depend on the initial condition; here $P(x,y,0)=\delta_{x,1}\delta_{y,1}$ [26].

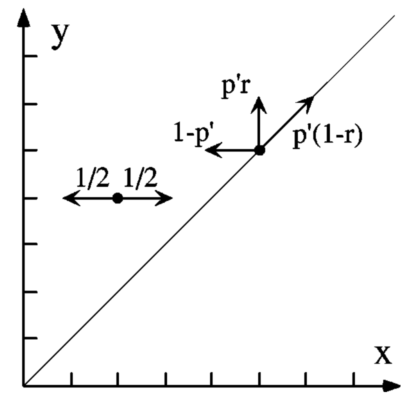

FIG. 1. Transition probabilities in the (x,y) plane.

Define the generating function $\hat{P}(x,y,z)=\Sigma_{t=0}^{\infty}z^tP(x,y,t)$ (and similarly for D, etc.). $\hat{P}$ satisfies

$$\begin{gathered} z^{-1}\hat{P}(x,y)=\tfrac{1}{2}\hat{P}(x-1,y)+\tfrac{1}{2}\hat{P}(x+1,y), \\ x=1,2,\ldots,y-2; \quad y\geqslant 3. \end{gathered} \tag{4}$$

(we drop the argument z for brevity), subject to the boundary conditions

$$\hat{P}(0,y)=0, \tag{5}$$

$$z^{-1}\hat{P}(y-1,y)=\tfrac{1}{2}\hat{P}(y-2,y)+p'r\hat{D}(y-1)+(1-p')\hat{D}(y), \tag{6}$$

$$z^{-1}\hat{D}(y)=\tfrac{1}{2}\hat{P}(y-1,y)+p'(1-r)\hat{D}(y-1). \tag{7}$$

The solution of Eqs. (4) and (5) is

$$\hat{P}(x,y)=\hat{C}(y)(\lambda_+^x-\lambda_-^x); \quad \lambda_\pm=z^{-1}\pm\sqrt{z^{-2}-1}, \tag{8}$$

with $\hat{C}(y)$ yet to be determined. Noting that $\lambda_+=\lambda_-^{-1}\equiv\lambda$, we define $\lambda(x)\equiv\lambda^x-\lambda^{-x}$; thus $\hat{P}(x,y)=\hat{C}(y)\lambda(x)$. Note also the recurrence relation for integer x: $\lambda(x-1)-2z^{-1}\lambda(x)+\lambda(x+1)=0$.

From Eq. (7) we find

$$\begin{gathered} \hat{D}(y)=\omega^{y-2}\hat{D}(2)+\tfrac{1}{2}z\omega^y\sum_{y'=3}^{y}\omega^{-y'}\hat{P}(y'-1,y'), \\ \omega=p'(1-r)z. \end{gathered} \tag{9}$$

It follows that $\hat{D}(y)-\omega\hat{D}(y-1)=(z/2)\hat{C}(y)\lambda(y-1)$. Therefore, subtracting from Eq. (5) ω times the corresponding equation for $y-1$, and using the recursion relation for $\lambda(x)$, we find

$$\frac{\hat{C}(y)}{\hat{C}(y-1)}=p'\frac{r\lambda(y-2)+(1-r)\lambda(y-1)}{(p'-\frac{1}{2})\lambda(y-1)+\frac{1}{2}\lambda(y+1)}, \tag{10}$$

which, noting that $\hat{C}(2)=\hat{P}(1,2)/\lambda(1)$, provides the complete solution for the probability generating function.

To determine the survival probability $S(t) = \Sigma_{y=0}^{\infty}\Sigma_{x=0}^{y}P(x,y,t)$ as $t\to\infty$, we analyze the singular behavior of $\hat{S}(z)$ as $z\to 1$. In this limit $\lambda = 1+\sqrt{2\epsilon}+\mathcal{O}(\epsilon)$, where $\epsilon = 1-z$, and we find the dominant term to be [22]

$$\hat{S}\sim\frac{1}{\sqrt{2(1-z)}}\sum_{y=3}^{\infty}\hat{C}(y)[\lambda(y/2)]^2, \tag{11}$$

which converges for $z<1$. Note that $\lambda(n)\sim 2n\sqrt{2\epsilon}$ for $n\epsilon \ll 1$, while in the opposite limit $\lambda(n)\sim\lambda^n$. For $n\epsilon\ll 1$ we may thus use the $z\to 1$ limiting expression for $\hat{C}(n)$, that is,

$$\hat{C}_0(y) = \frac{\hat{P}(1,2)}{2\sqrt{2(1-z)}}\frac{\Gamma(2+1/p')}{\Gamma(2-r)}\frac{\Gamma(y-r)}{\Gamma(y+1/p')}. \tag{12}$$

We then approximate the summand in Eq. (11) by

$$f_n = \begin{cases} 2n^2\hat{C}_0(n)\epsilon, & n\epsilon\leqslant 1,\\ 2N^2\hat{C}_0(N)\epsilon a^{n-N}\lambda^{n-N}, & n\epsilon>1, \end{cases} \tag{13}$$

where $N = 1/\sqrt{\epsilon}$ is rounded to an integer, and

$$a = p'\frac{r+(1-r)\lambda}{(p'-1/2)\lambda+(1/2)\lambda^3}. \tag{14}$$

Since $\hat{S}$ and $F(z)\equiv\Sigma_{n=0}^{\infty}f_n$ have the same radius of convergence and $\lim_{n\to\infty}\hat{C}(n)[\lambda(n/2)]^2/f_n$ is finite, an Abelian theorem implies that the singular behavior of $\hat{S}$ is the same as that of $F(z)/\sqrt{\epsilon}$ [4]. Write $F(z)/\sqrt{\epsilon} = 2\epsilon^{1/2}\Sigma_{n=0}^{N}n^2\hat{C}_0(n) + 2\epsilon^{1/2}N^2\hat{C}_0(N)\Sigma_{n=N+1}^{\infty}(\lambda a)^{n-N}$. We approximate the first sum by an integral; using $\hat{C}_0(n)\sim\epsilon^{-1/2}n^{-r-1/p'}$ we then find that the first term $\sim\epsilon^{(r+1/p'-3)/2}$. The same holds for the second term when we note that $1-a\lambda\sim\epsilon^{1/2}$ and $\hat{C}_0(N) \sim\epsilon^{-(1-r-1/p')/2}$. Thus $\hat{S}\sim(1-z)^{\delta-1}$ (plus less singular terms) as $z\to 1$, where $\delta = (r+1/p'-1)/2$. Finally, noting that for large n, the coefficient of z^n in $(1-z)^{\delta-1}$ is $n^{-\delta}/\Gamma(1-\delta)$, we have that the survival probability decays asymptotically $\sim t^{-\delta}$; in particular, for $p' = 1/2$ we have $S(t)\sim t^{-(1+r)/2}$. Note also that the larger p' is, the slower the decay of the survival probability, as expected.

While the amplitude of the leading term in $S(t)$ can be evaluated exactly for $p' = 1/2$ and $r = 0$ or 1 [22], we have not found a closed-form expression in the general case. Numerical evaluation of the exact expression for $\hat{S}$ does, however, lead to an amplitude in perfect agreement with that found in the transition matrix analysis.

We turn now to results obtained via numerical iteration of the transition matrix defined in Eq. (2). We studied the process to a maximum time of up to 3×10^5, quite sufficient to observe asymptotic behavior. In Fig. 2 we show the survival probability for $p' = 1/2$ and various values of r. In each case there is a power-law decay, with the decay exponent δ varying continuously with r. [The plateaus in $S(t)$ at short times reflect that for $r = 0$ the walker can only reach the origin on odd-numbered steps. Similarly, for $r = 1$ the walker cannot reach the origin on the third step]. We verify that the mean position $\langle x_t\rangle\sim t^{1/2}$ (and that $\langle x_t^2\rangle\sim t$; the averages are over surviving walks) independent of r. The amplitudes for the displacement and its second moment do decrease with increasing r.

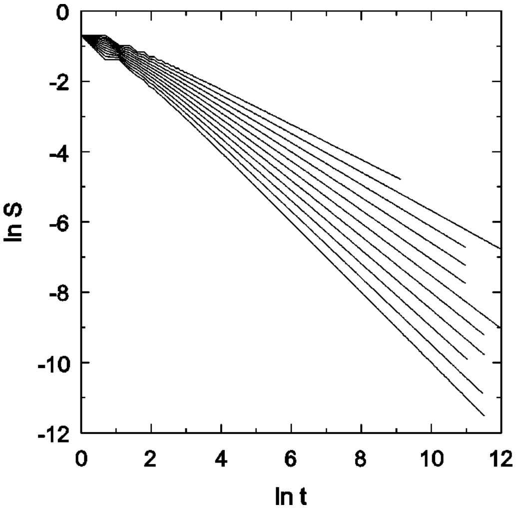

FIG. 2. Survival probability versus time for random walk with a movable partial reflector, $p' = 1/2$. Reflection probability (top to bottom) $r = 0, 0.1, \ldots, 1$.

To determine the decay exponent precisely, we study the local slope $\delta(t)$, given by a least-squares linear fit to the $\ln S$ data for a set of 11 equally spaced values (increments of 0.05) of $\ln t$. Plots of $\delta(t)$ versus t^{-1} lead to asymptotic ($t\to\infty$) values confirming $\delta = (1+r)/2$ to better than one part in 2000. It is possible to improve the analysis further by determining the leading correction to scaling in an expansion of the form: $S(t)\simeq At^{-\delta}[1-Bt^{-\Delta_1}+\cdots]$. For $r = 0$ we find $A = 0.7979$, $B = 0.78$, and $\Delta_1 = 1.00$, in accord with the exact expansion $P(t)\simeq\sqrt{2/(\pi t)}[1-(3/4)t^{-1}+\ldots]$. For $r = 1$ our data yield $A = 1.0000$ (in agreement with the exact result), $B = 1.332$, and $\Delta_1 = 1.00$. In these two cases, plotting

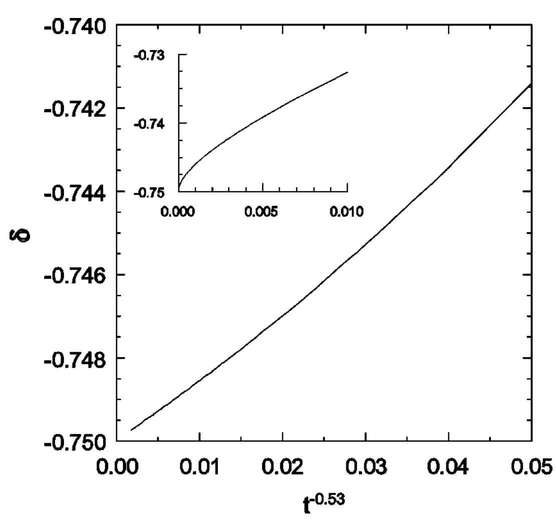

FIG. 3. Local slope $\delta(t)$ for $p' = r = 1/2$. The inset shows the same data plotted versus $1/t$.

$\delta(t)$ versus $t^{-\Delta_1}=t^{-1}$, we obtain $\delta=0.5000$ and 0.999985, respectively. For r in the interval [0.1,0.8] we find a much smaller correction to scaling exponent, $\Delta_1 \simeq 0.54$, while for $r=0.9$, $\Delta_1 \simeq 0.62$. Plotting $\delta(t)$ versus $t^{-\Delta_1}$ (see Fig. 3), we obtain exponents that agree with the theoretical value to better than one part in 2×10^4. The attractive simplification, $\Delta_1=1/2$ for $r\neq 0$, 1, yields essentially the same values for δ, and is in fact to be expected, given that generic corrections to $\hat{S}$ should be $\propto(1-z)^{\delta-1+m/2}$ $(m=1,2,\ldots)$.

The case of compact directed percolation is somewhat more complicated than the random-walk problem analyzed above, as we must now keep track of three variables, viz., the distance X between the two walkers, and the respective distances between the walkers and the associated reflectors [23]. Simulation results for CDP with movable partial reflectors again show δ varying continuously with r, but over a somewhat wider range: for $r=1$, $\delta=1.158(3)$.

In one-dimensional spreading models (contact process [10], pair contact process [16], etc.) the survival and spread of activity may again be described in terms of the position of two points that bound the active region, as in CDP. But here the boundaries of the active region do not follow simple random walks: their dynamics involves a variable step size (due to gaps in the distribution of active sites), and a significant memory (the step-size distribution depends on the history of the previous step directions). It seems reasonable, nevertheless, to expect that varying the probability for advancing into virgin territory will change the scaling of the survival probability, just as observed here. A detailed investigation of this issue is a high priority for future work.

In summary, we have uncovered the rather remarkable property of a continuously varying critical exponent governing the survival probability of a random walk with an absorbing boundary, in the presence of a movable partial reflector. Our results suggest an alternative approach to understanding nonuniversal spreading in models with an infinite number of absorbing configurations.

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